

Chapter 10

Introduction to Many-Body Perturbation Theory

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We introduce the language of second quantization in the context of quantum many body systems and treat the weakly interacting Bose gas at low temperatures.

10.1 Introduction

As we saw in Chapter 8, it is difficult to treat the interparticle interactions in a classical many body system of particles. As might be expected, the analysis of the analogous quantum system is even more difficult.

Just as we developed the density expansion of a classical gas by doing perturbation theory about the ideal gas, we will first treat an interacting many-body quantum system by starting from the single particle approximation. We know that the wave function $\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$ for a system of N identical interacting particles can be expanded in terms of the wave function $\Phi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$ of the noninteracting system. The wave function Φ is given in terms of suitably symmetrized products of the single particle eigenfunctions $\phi(\mathbf{r}_i)$. If we adopt periodic boundary conditions, $\phi_{\mathbf{k}}(\mathbf{r})$ is given by

$$\phi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{L^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (10.1)$$

where L is the linear dimension of the system. Note that ϕ is an eigenfunction of the momentum $\mathbf{p} = \hbar\mathbf{k}$.

If the particles are bosons, the wave function Ψ and hence Φ must be symmetric with respect to the interchange of any two particles. If the particles are fermions, Ψ and Φ must be antisymmetric with respect to the interchange of any two particles. The latter condition is the generalization of the Pauli exclusion principle.

Because of the impossibility of distinguishing identical particles, it is useful to describe non-interacting quantum systems by specifying only the number of particles in each single particle state (see Section 6.5). That is, instead of working in coordinate space, we can represent the basis functions of the many-body wave functions by

$$|n_1 n_2 \dots\rangle, \quad (10.2)$$

where n_k is the number of particles in the single particle state ϕ_k . For fermions n_k equals 0 or 1; there is no restriction for bosons. For a system with a fixed number of particles N , the occupation numbers n_k satisfy the condition

$$N = \sum_k n_k. \quad (10.3)$$

We also learned in Section 6.5 that it is convenient to treat quantum mechanical systems in the grand canonical ensemble in which the number of particles in a particular single particle quantum state may vary. For this reason we next introduce a formalism that explicitly allows us to write the energy of the system in terms of operators that change the number of particles in a given state.

10.2 Occupation Number Representation

If we specify a state of the system in the occupation number representation, it is convenient to introduce the operators \hat{a}_k and \hat{a}_k^\dagger that act on states such as in (10.2). For bosons we define \hat{a}_k and \hat{a}_k^\dagger by

$$\hat{a}_k |\dots n_k \dots\rangle = \sqrt{n_k} |\dots n_k - 1 \dots\rangle, \quad (10.4a)$$

and

$$\hat{a}_k^\dagger |\dots n_k \dots\rangle = \sqrt{n_k + 1} |\dots n_k + 1 \dots\rangle \quad (10.4b)$$

From the definition (10.4a) we see that \hat{a}_k reduces the number of particles in state k and leaves the other occupation numbers unchanged. For this reason \hat{a}_k is called the *annihilation* or *destruction operator*. Similarly, from (10.4b) we see that \hat{a}_k^\dagger increases the occupation number of state k by unity and is called the *creation operator*. The factor of $\sqrt{n_k}$ is included in (10.4a) to normalize the N and $N - 1$ particle wave functions and to make the definitions consistent with the assertion that \hat{a}_k and \hat{a}_k^\dagger are Hermitian conjugates. The factor $\sqrt{1 + n_k}$ is included for the latter reason.

From the definitions in (10.4), it is easy to show that

$$\hat{a}_k \hat{a}_k^\dagger |n_k\rangle = (n_k + 1) |n_k\rangle \quad (10.5a)$$

$$\hat{a}_k^\dagger \hat{a}_k |n_k\rangle = n_k |n_k\rangle. \quad (10.5b)$$

We have written $|n_k\rangle$ for $|\dots, n_k, \dots\rangle$. By subtracting (10.5b) from (10.5a), we have

$$(\hat{a}_k \hat{a}_k^\dagger - \hat{a}_k^\dagger \hat{a}_k) |n_k\rangle = |n_k\rangle. \quad (10.6)$$

In general, we may write that

$$[\hat{a}_k, \hat{a}_k^\dagger] \equiv \hat{a}_k \hat{a}_k^\dagger - \hat{a}_k^\dagger \hat{a}_k = 1, \quad (10.7)$$

and show that

$$[\hat{a}_k, \hat{a}_k^\dagger] = \delta_{kk'}, \quad (10.8)$$

and

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0. \quad (10.9)$$

The commutation relations (10.8) and (10.9) define the creation and destruction operators \hat{a}_k^\dagger and \hat{a}_k .

The appropriate definition of \hat{a}_k and \hat{a}_k^\dagger is a little more tedious for fermions, and we shall simply define them by the *anticommutation* relations:

$$\{\hat{a}_k, \hat{a}_k^\dagger\} \equiv \hat{a}_k \hat{a}_k^\dagger + \hat{a}_k^\dagger \hat{a}_k = 1, \quad (10.10)$$

and

$$\{\hat{a}_k, \hat{a}_{k'}\} = \{\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger\} = 0. \quad (10.11)$$

Equation (10.11) is equivalent to the statement that it is not possible to create two particles in the same single particle state.

10.3 Operators in the Second Quantization Formalism

It is easy to show that for both Bose and Fermi statistics, the number operator \hat{N}_k is given by

$$\hat{N}_k = \hat{a}_k^\dagger \hat{a}_k. \quad (10.12)$$

The eigenvalues of \hat{N}_k acting on $|n_k\rangle$ are zero or unity for fermions and either zero or any positive integer for bosons.

We now wish to write other operators in terms of \hat{a}_k and \hat{a}_k^\dagger . To do so, we note that \hat{a}_k^\dagger and \hat{a}_k are the creation and destruction operators for a free particle with momentum $\mathbf{p} = \hbar\mathbf{k}$ described by the wave function (10.1). The kinetic energy is an example of a one-particle operator

$$\hat{T} = -\frac{\hbar^2}{2m} \sum_{i=1}^N \hat{\nabla}_i^2. \quad (10.13)$$

The form (10.13) in which the momentum \mathbf{p} is expressed as an operator is an example of what is called first quantization. Note that the sum in (10.13) is over the indistinguishable particles in the system. A more convenient form for \hat{T} in the *second quantization* formalism is given by

$$\hat{T} = \sum_{\mathbf{p}} \epsilon_p \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}, \quad (10.14)$$

where $\epsilon_p = p^2/2m$ and $\mathbf{p} = \hbar\mathbf{k}$. Note that the kinetic energy is diagonal in \mathbf{p} . The form of (10.14) is suggestive and can be interpreted as the sum of the kinetic energy in state \mathbf{p} times the number of particles in this state.

The form of the two-particle potential energy operator \hat{U} can be obtained from straightforward but tedious arguments. The result can be written as

$$\hat{U} = \frac{1}{2} \sum_{\mathbf{k}'_1, \mathbf{k}'_2, \mathbf{k}_1, \mathbf{k}_2} \langle \mathbf{k}'_1 \mathbf{k}'_2 | u | \mathbf{k}_1 \mathbf{k}_2 \rangle \hat{a}_{\mathbf{k}'_1}^\dagger \hat{a}_{\mathbf{k}'_2}^\dagger \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2}. \quad (10.15)$$

The summation in (10.15) is over all values of the momenta (wave vectors) of a pair of particles such that the total momentum is conserved in the interaction:

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}'_1 + \mathbf{k}'_2. \quad (10.16)$$

The matrix element $\langle \mathbf{k}'_1 \mathbf{k}'_2 | u | \mathbf{k}_1 \mathbf{k}_2 \rangle$ is given by

$$\langle \mathbf{k}'_1 \mathbf{k}'_2 | u | \mathbf{k}_1 \mathbf{k}_2 \rangle = \frac{1}{V^2} \iint e^{i(\mathbf{k}'_1 - \mathbf{k}_1) \cdot \mathbf{r}_1 + i(\mathbf{k}'_2 - \mathbf{k}_2) \cdot \mathbf{r}_2} u(|\mathbf{r}_2 - \mathbf{r}_1|) d\mathbf{r}_1 d\mathbf{r}_2. \quad (10.17)$$

We next make the change of variables, $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, and write

$$\langle \mathbf{k}'_1 \mathbf{k}'_2 | u | \mathbf{k}_1 \mathbf{k}_2 \rangle = \frac{1}{V^2} \iint e^{i(\mathbf{k}_1 - \mathbf{k}'_1 + \mathbf{k}_2 - \mathbf{k}'_2) \cdot \mathbf{R}} e^{i(\mathbf{k}_1 - \mathbf{k}'_1 - \mathbf{k}_2 + \mathbf{k}'_2) \cdot \mathbf{r}/2} u(r) d\mathbf{R} d\mathbf{r}. \quad (10.18a)$$

Because of the homogeneity of space, the integral over \mathbf{R} can be done yielding a Dirac delta function and the condition (10.16). We thus obtain

$$\langle \mathbf{k}'_1 \mathbf{k}'_2 | u | \mathbf{k}_1 \mathbf{k}_2 \rangle = u(k) = \int e^{-i\mathbf{k} \cdot \mathbf{r}} u(r) d\mathbf{r}, \quad (10.18b)$$

where $\mathbf{k} = \mathbf{k}'_2 - \mathbf{k}_2 = -(\mathbf{k}'_1 - \mathbf{k}_1)$ is the momentum (wave vector) transferred in the interaction.

With these considerations we can write the Hamiltonian in the form

$$\hat{H} = \sum_p \frac{p^2}{2m} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{1}{2V} \sum_{\mathbf{k}, \mathbf{p}_1, \mathbf{p}_2} u(k) \hat{a}_{\mathbf{p}_1 + \mathbf{k}}^\dagger \hat{a}_{\mathbf{p}_2 - \mathbf{k}}^\dagger \hat{a}_{\mathbf{p}_2} \hat{a}_{\mathbf{p}_1}. \quad (10.19)$$

We have written \mathbf{p}_1 and \mathbf{p}_2 instead of \mathbf{k}_1 and \mathbf{k}_2 in (10.19) and chosen units such that $\hbar = 1$. The order of the operators in (10.15) and (10.19) is important for fermions because the fermion operators anticommute. The order is unimportant for bosons. The form of the interaction term in (10.19) can be represented as in Figure 10.1.

10.4 Weakly Interacting Bose Gas

A calculation of the properties of the dilute Bose gas was once considered to have no direct physical relevance because the gases that exist in nature condense at low temperatures. However, such a calculation was interesting because the properties of the weakly interacting Bose gas are similar to liquid ^4He . In particular, a dilute Bose gas can be a superfluid even though an ideal Bose gas cannot. Moreover, in recent years, the dilute Bose gas at low temperatures has been created in the laboratory (see [references](#)).

The condition for a gas to be dilute is that the range of interaction σ should be small in comparison to the mean distance between the particles, $\rho^{-1/3}$, that is $\rho\sigma^3 \ll 1$. Because the gas is

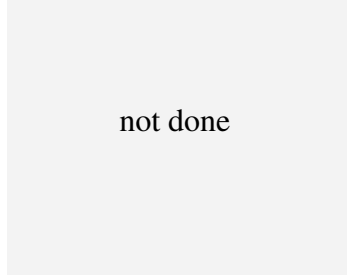


Figure 10.1: Representation of the matrix element in (10.15).

dilute, we need to consider only binary interactions between particles using quantum perturbation theory. The difficulty is that because of the rapid increase in the interparticle potential $u(r)$ at small r , ordinary perturbation theory (the Born approximation) cannot be directly applied.

We can circumvent the lack of applicability of the Born approximation by the following argument. The scattering cross section is given by $|f|^2$, where f is the scattering amplitude. In the Born approximation, f is given by

$$f(k) = -\frac{m}{4\pi\hbar^2} \int u(r) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}, \quad (10.20)$$

where $\hbar\mathbf{k}$ is the momentum transferred in the interaction. In the limit of low temperatures, the particle momenta are small, and we can set $\mathbf{k} = 0$ in (10.20). If we set $f(k=0) = -a$, where a is the scattering amplitude, we obtain

$$a = mU_0/4\pi\hbar^2 \quad (10.21)$$

where

$$U_0 = \int u(r) d\mathbf{r}. \quad (10.22)$$

In the following, we will set $u(\mathbf{k}=0) = U_0 = 4\pi\hbar^2 a/m$, so that we will be able to mimic the result of doing a true perturbation theory calculation.¹

If we assume that $u(k) = U_0$ for all k , a constant, we can write the Hamiltonian as

$$\hat{H} = \sum_{\mathbf{p}} \frac{p^2}{2m} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{U_0}{2V} \sum_{\mathbf{k}, \mathbf{p}_1, \mathbf{p}_2} \hat{a}_{\mathbf{p}_1 - \mathbf{k}}^\dagger \hat{a}_{\mathbf{p}_2 + \mathbf{k}}^\dagger \hat{a}_{\mathbf{p}_2} \hat{a}_{\mathbf{p}_1}. \quad (10.23)$$

The form of (10.23) is the same for Bose or Fermi statistics. Only the commutation relations for the creation and destruction operators are different.

We now follow the approximation method developed by Bogolyubov (1947). In the limit $U_0 \rightarrow 0$, \hat{H} reduces to the Hamiltonian of the ideal Bose gas. We know that the latter has

¹In the language of quantum mechanics, we need to replace the bare interaction u by the t matrix. This replacement is the quantum mechanical generalization of replacing $-\beta u$ by the Mayer f function. Not surprisingly, this replacement can be represented by an infinite sum of ladder-type diagrams. Note that if we interpret the Mayer f function as the effective interaction between particles, the first cumulant in a high temperature expansion would yield the same result as the first term in the classical virial expansion.

a condensate, that is, there is macroscopic occupation of the zero momentum state, so that at $T = 0$, $N_0 = N$, and $N_{\mathbf{p}} = 0$ for $\mathbf{p} \neq 0$. For the weakly interacting Bose gas, we expect that the low lying states do not have zero occupation, but that $N_{\mathbf{p}}$ for $\mathbf{p} > 0$ is small so that $N_0 \approx N$. We proceed by assuming that $N - N_0$ is small and extract the $\mathbf{k} = 0$ terms in \hat{H} . For example,

$$\hat{N} = \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} = \hat{a}_0^\dagger \hat{a}_0 + \sum_{\mathbf{p} \neq 0} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}. \quad (10.24)$$

Because $\hat{a}_0^\dagger \hat{a}_0 = N_0 \approx N$ is much larger than unity, it follows that $\hat{a}_0 \hat{a}_0^\dagger - \hat{a}_0^\dagger \hat{a}_0 = 1$ is small in comparison to \hat{a}_0 and \hat{a}_0^\dagger and hence \hat{a}_0 and \hat{a}_0^\dagger may be regarded as numbers (equal to $\sqrt{N_0}$), and we can ignore the fact that they do not commute.

We now expand the potential energy in (10.23) in powers of the small quantities $\hat{a}_{\mathbf{p}}$, $\hat{a}_{\mathbf{p}}^\dagger$ for $\mathbf{p} \neq 0$. The zeroth-order term is

$$\frac{U_0}{2V} a_0^\dagger a_0^\dagger a_0 a_0 = \frac{U_0}{2V} a_0^4 = \frac{U_0}{2V} N_0^2. \quad (10.25)$$

There are no first-order terms proportional to a_0^3 , because they cannot be made to satisfy conservation of momentum. The second-order contributions are proportional to $(U_0/2V)N_0$ and are given by

(a)	$\mathbf{p}_1 = \mathbf{p}_2 = 0, \mathbf{k} \neq 0$	$\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger$
(b)	$\mathbf{k} = -\mathbf{p}_1, \mathbf{p}_2 = 0$	$\hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_1}$
(c)	$\mathbf{k} = \mathbf{p}_2, \mathbf{p}_1 = 0$	$\hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{p}_2}$
(d)	$\mathbf{p}_1 = -\mathbf{p}_2 = -\mathbf{k}$	$\hat{a}_{\mathbf{p}_1} \hat{a}_{-\mathbf{p}_1}$
(e)	$\mathbf{k} = \mathbf{p}_1 = 0, \mathbf{p}_2 \neq 0$	$\hat{a}_{\mathbf{p}_2}^\dagger \hat{a}_{\mathbf{p}_2}$
(f)	$\mathbf{k} = \mathbf{p}_2 = 0, \mathbf{p}_1 \neq 0$	$\hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_1}$

We will ignore all higher order terms, which is equivalent to ignoring the interaction between excited particles. Hence, if we extend the above approximations to T above T_c , our approximate Hamiltonian would reduce to the Hamiltonian for the ideal gas.

The approximate Hamiltonian can now be written as

$$\hat{H} = \sum'_{\mathbf{p}} \frac{p^2}{2m} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{U_0}{2V} N_0^2 + \frac{U_0}{2V} N_0 \sum'_{\mathbf{k}} [\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger + \hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} + 4\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}]. \quad (10.26)$$

The notation \sum' denotes that the sum excludes terms with $\mathbf{p} = 0$ and $\mathbf{k} = 0$.

In general, we have

$$N = a_0^2 + \sum'_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} = N_0 + \sum'_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}. \quad (10.27)$$

For consistency, we replace N_0^2 in (10.26) by $N_0^2 = N^2 - 2N \sum'_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}$. Similarly N_0 in (10.26) may be replaced by N . The result of these replacements is that

$$\hat{H} \approx \hat{H}_B = \frac{N^2}{2V} U_0 + \sum'_{\mathbf{p}} \epsilon_p \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{N}{2V} U_0 \sum'_{\mathbf{k}} [\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger + \hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} + 2\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}]. \quad (10.28)$$

Note that H_B only allows excitation of pairs of momentum \mathbf{k} and $-\mathbf{k}$ from the condensate and re-entry of such pairs into the condensate.

The approximate Hamiltonian \hat{H}_B is bilinear in \hat{a} and \hat{a}^\dagger . This form is similar to that of a harmonic oscillator. This similarity suggests that we can diagonalize \hat{H}_B by making an appropriate linear transformation of the operators \hat{a} and \hat{a}^\dagger . If \hat{H}_B is put into diagonal form, then \hat{H}_B would have the same form as an ideal gas, and we could easily calculate the energy eigenvalues.

We define new operators \hat{b}^\dagger and \hat{b} by

$$\hat{a}_{\mathbf{k}} = u_k \hat{b}_{\mathbf{k}} + v_k \hat{b}_{-\mathbf{k}}^\dagger \quad (10.29a)$$

$$\hat{a}_{\mathbf{k}}^\dagger = u_k \hat{b}_{\mathbf{k}}^\dagger + v_k \hat{b}_{-\mathbf{k}}, \quad (10.29b)$$

and require them to satisfy the Bose commutation relations

$$\hat{b}_{\mathbf{k}} \hat{b}_{\mathbf{k}'}^\dagger - \hat{b}_{\mathbf{k}'}^\dagger \hat{b}_{\mathbf{k}} = \delta_{\mathbf{k}\mathbf{k}'} \quad \text{and} \quad \hat{b}_{\mathbf{k}} \hat{b}_{\mathbf{k}'} = \hat{b}_{\mathbf{k}'} \hat{b}_{\mathbf{k}}. \quad (10.30)$$

As shown in Problem 10.1, \hat{b}^\dagger and \hat{b} satisfy the Bose commutation relations only if the relation (10.31) between u_k and v_k is satisfied:

$$u_k^2 - v_k^2 = 1. \quad (10.31)$$

Problem 10.1. (a) Use (10.29) to express \hat{b}^\dagger and \hat{b} in terms of \hat{a}^\dagger and \hat{a} . (b) Show that the commutation relations (10.30) are satisfied only if (10.31) is satisfied.

If we substitute the above expressions for \hat{a}^\dagger and \hat{a} into (10.28), we obtain

$$\hat{H}_B = E_0 + \hat{H}_D + \hat{H}_I \quad (10.32a)$$

where

$$E_0 = \frac{N^2 U_0}{2V} + \sum_{\mathbf{k}}' \left[(\epsilon_p + \frac{NU_0}{V}) v_k^2 + \frac{NU_0}{V} u_k v_k \right] \quad (10.32b)$$

$$\hat{H}_D = \sum_{\mathbf{k}}' \left[(\epsilon_p + \frac{NU_0}{V}) (u_k^2 + v_k^2) - \frac{2NU_0}{V} u_k v_k \right] \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} \quad (10.32c)$$

$$\hat{H}_I = \sum_{\mathbf{k}}' \left[(\epsilon_k + \frac{NU_0}{V}) (u_k v_k + \frac{1}{2} v_k (u_k^2 + v_k^2)) \right] (\hat{b}_{\mathbf{k}}^\dagger \hat{b}_{-\mathbf{k}}^\dagger + \hat{b}_{\mathbf{k}} \hat{b}_{-\mathbf{k}}). \quad (10.32d)$$

From the form of (10.32), we see that \hat{H}_B would be diagonal if $\hat{H}_I = 0$. This condition is satisfied if

$$2(\epsilon_k + \frac{NU_0}{V}) (u_k v_k + v_k (u_k^2 + v_k^2)) = 0, \quad (10.33)$$

and the relation (10.31) is satisfied. Note that we have two equations for the two unknown u_k and v_k . We can satisfy the relation (10.31) automatically by letting

$$u_k = \cosh \theta_k \quad (10.34a)$$

$$v_k = \sinh \theta_k. \quad (10.34b)$$

If we use the identities $2u_k v_k = 2 \cosh \theta_k \sinh \theta_k = \sinh 2\theta_k$ and $u_k^2 + v_k^2 = \cosh 2\theta_k$, we can express (10.33) as

$$\left(\epsilon_k + \frac{NU_0}{V}\right) \sinh 2\theta_k + \frac{NU_0}{V} \cosh 2\theta_k = 0, \quad (10.35)$$

or

$$\tanh 2\theta_k = -\frac{\rho U_0}{\epsilon_k + \rho U_0}. \quad (10.36)$$

Note that (10.36) has a solution for all k only if $U_0 > 0$.

The solution (10.36) is equivalent to

$$u_k^2 + v_k^2 = \frac{\epsilon_k + \rho U_0}{E(k)} \quad (10.37)$$

and

$$2u_k v_k = -\frac{\rho U_0}{E(k)}, \quad (10.38)$$

where

$$E(k) = \sqrt{\epsilon_k(\epsilon_k + 2\rho U_0)}. \quad (10.39)$$

$$u_k^2 = \frac{1}{2} \left[\frac{\epsilon_k + \rho U_0}{E(k)} + 1 \right], \quad (10.40a)$$

$$v_k^2 = \frac{1}{2} \left[\frac{\epsilon_k + \rho U_0}{E(k)} - 1 \right]. \quad (10.40b)$$

If we substitute u_k and v_k into \hat{H}_B , we obtain

$$\hat{H}_B = \frac{1}{2} N \rho U_0 + \sum_{\mathbf{k}}' [E(k) - \epsilon_k - \rho U_0] + \sum_{\mathbf{k}}' E(k) \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}}. \quad (10.41)$$

From the form of (10.41) we see that $\hat{b}_{\mathbf{k}}^\dagger$ and $\hat{b}_{\mathbf{k}}$ are the creation and destruction operators for quasiparticles or elementary excitations with energy $E(k)$ obeying Bose statistics. If we replace U_0 by $4\pi\hbar^2 a/m$, we see that the quasiparticle energy is given by

$$E(p) = \sqrt{c^2 p^2 + (p^2/2m)^2}, \quad (10.42)$$

where

$$c = \sqrt{\frac{4\pi\hbar^2 \rho a}{m^2}}. \quad (10.43)$$

Note that for small p , $E(p)$ is proportional to p and hence the excitations are phonons with velocity c .

The ground state energy E_0 is given by

$$E_0 = \frac{1}{2} N \rho U_0 + \sum_{\mathbf{k}}' [E(k) - \epsilon_k - \rho U_0]. \quad (10.44)$$

We can replace the summation over discrete values of \mathbf{k} by an integration over \mathbf{p} and multiply by $V/(2\pi\hbar)^3$. We obtain (see Huang)

$$\frac{E_0}{N} = \frac{2\pi a\rho}{m} \left[1 + \frac{128}{15} \sqrt{\frac{a^3\rho}{\pi}} \right]. \quad (10.45)$$

Problem 10.2. Show that c is equal to the sound speed using the relation (see Reif)

$$c = (\rho\kappa_S)^{-1/2}, \quad (10.46)$$

where κ_S is the adiabatic compressibility:

$$\kappa_S = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_S. \quad (10.47)$$

The above relations can be used to express c as

$$c^2 = \left(\frac{\partial \rho}{\partial P} \right)_S. \quad (10.48)$$

At $T = 0$, the pressure is given by

$$P = -\frac{\partial E_0}{\partial V}. \quad (10.49)$$

Use the above relations and (10.45) to show that the calculated speed of sound is consistent with the phonon speed (10.43) to lowest order in $(\rho a^3)^{1/2}$.

Problem 10.3. The number of quasiparticles of momentum \mathbf{p} for $T > 0$ is given by

$$\bar{n}_{\mathbf{p}} = \frac{1}{e^{\beta E(p)} - 1}. \quad (10.50)$$

Why is the chemical potential equal to zero?

Problem 10.4. The momentum distribution of the actual particles in the gas is given by

$$\bar{N}_{\mathbf{p}} = \overline{\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}}. \quad (10.51)$$

Use the relation between $\hat{a}_{\mathbf{p}}^\dagger$ and $\hat{a}_{\mathbf{p}}$ and $\hat{b}_{\mathbf{p}}^\dagger$ and $\hat{b}_{\mathbf{p}}$, and the fact that the products $\hat{b}_{\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}}^\dagger$ and $\hat{b}_{-\mathbf{p}} \hat{b}_{\mathbf{p}}$ have no diagonal matrix elements to show that

$$\bar{N}_{\mathbf{p}} = \frac{\bar{n}_{\mathbf{p}} + f_{\mathbf{p}}(\bar{n}_{\mathbf{p}} + 1)}{1 - f_{\mathbf{p}}}, \quad (10.52)$$

where

$$f_{\mathbf{p}} = \frac{m}{4\pi a\rho\hbar^2} \left[E(p) - \frac{p^2}{2m} - mc^2 \right]. \quad (10.53)$$

This result is valid only for $\mathbf{p} \neq 0$. At $T = 0$, $\bar{n}_{\mathbf{p}} = 0$ for $\mathbf{p} \neq 0$. Show that

$$\bar{N}_p = \frac{m^2 c^4}{2E(p)[E(p) + p^2/2m + mc^2]}. \quad (10.54)$$

The number of particles with zero momentum is

$$\bar{N}_0 = 1 - \sum_{\mathbf{p}}' \bar{N}_{\mathbf{p}} = 1 - V \int \frac{d^3p}{(2\pi\hbar)^3} \bar{N}_{\mathbf{p}}. \quad (10.55)$$

Note that the interaction between the particles causes the appearance of particles with nonzero momentum even at $T = 0$. Use (10.54) to show that

$$\frac{\bar{N}_0}{N} = 1 - \frac{8}{3} \left(\frac{\rho a^3}{\pi} \right)^{1/2}. \quad (10.56)$$